# Isolated singularities

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Def. $z \in \mathbb{C}$ is called an isolated singularity of a function $f$ if $f \in A(B(z,r) \setminus \{z\})$ for some $r>0$.

Case 1: removable singularity.

Theorem. The following are equivalent for an isolated singularity $z_0$ of $f$:

1) $\exists F \in A(B(z_0,r))$ : $F(z)=f(z)$ for $z \in B(z_0,r)$, $z \neq z_0$.

2) $\exists \lim\limits_{z\to z_0} f(z)$. (Equivalently: $\exists F$ - continuous on $B(z_0,r)$, $F(z)=f(z)$ for $z\in B(z_0,r)$, $z\neq z_0$)

3) $f$ is bounded near $z_0$:

$\exists M, \delta > 0 : \forall z \neq z_0$, $|z-z_0|<\delta \Rightarrow |f(z)|<M$

4) $\lim\limits_{z\to z_0} (z-z_0) f(z) = 0$.

Proof. 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) - obvious.

4) $\Rightarrow$ 1). Take $\delta < r$. $\forall z \in B(z_0,\delta)$, $z\neq z_0$, define a function $G$:

$$G(w) := \frac{f(w)-f(z)}{w-z} \in A(B(z_0,r)\setminus\{z,z_0\}),$$

$$\lim_{w\to z}(w-z)G(w)=0,$$

$$\lim_{w\to z_0}(w-z_0)G(w) = \lim_{w\to z_0}(w-z_0)\frac{f(w)-f(z)}{w-z} = \lim_{w\to z_0}\frac{(w-z_0)f(w)}{w-z} - \lim_{w\to z_0}\frac{(w-z_0)f(z)}{w-z} = 0 - \frac{(z_0-z_0)f(z)}{z_0-z} = 0.$$

So, by Cauchy,

$$\frac{1}{2\pi i}\oint_{C_\delta} G(w)\,dw = 0 \Rightarrow f(z) = \frac{1}{2\pi i}\oint_{C_\delta}\frac{f(w)}{w-z}\,dw, \quad z\in B(z_0,\delta)\setminus\{z_0\}$$

$C_\delta = \{z_0+\delta e^{it}\}$ - counterclockwise oriented circle.

Define
$$F(z) = \begin{cases} f(z), & z\in B(z_0,r)\setminus\{z_0\} \\ \frac{1}{2\pi i}\oint_{C_\delta}\frac{f(w)}{w-z}\,dw, & z\in B(z_0,\delta). \end{cases}$$

Then: $F$ is well-defined (two definitions agree on $B(z_0,\delta)\setminus\{z_0\}$), $F \in A(B(z_0,r)\setminus\{z_0\})$ $(=f(z))$

$F\in A(B(z_0,\delta))$ (including $z_0$ — it is Cauchy integral!).

So $F\in A(B(z_0,r))$, $F(z)=f(z)$, $z\neq z_0$ ∎

Case 1') Zeroes of analytic functions.

Theorem. Let $\Omega$ be a region, $f\in A(\Omega)$.

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Theorem. Let $\Omega$ be a region, $f \in \mathcal{A}(\Omega)$.

Assume $\exists z_0 \in \Omega: \forall n: f^{(n)}(z_0)=0$. Then $f(z) \equiv 0$.

Proof. Let

$\Omega_k := \{z \in \Omega : f^{(k)}(z) \neq 0\}$ - open set. ($z \in \Omega_k \Rightarrow \exists \delta>0: B(z,\delta) \subset \Omega_k$)

$\Omega^1 := \bigcup \Omega_k = \{z \in \Omega : \exists k : f^{(k)}(z) \neq 0\}$ - open (union of open sets).

$\Omega^2 := \Omega \setminus \Omega^1 = \{z \in \Omega: \forall k: f^{(k)}(z)=0\}$.

$z \in \Omega^2 \Rightarrow \forall w \in B(z,r),\ f(w) = \sum \frac{f^{(k)}(z)}{k!}(w-z)^k = 0 \Rightarrow f \equiv 0$ in $B(z,r) \Rightarrow B(z,r) \subset \Omega_2$.

$r := \mathrm{dist}(z, \partial\Omega)$

So $\Omega^2$ is also open. $\Omega$ is connected, so either $\Omega^1 = \emptyset$ (and then $f \equiv 0$ in $\Omega$) or $\Omega^2 = \emptyset$ (so $\forall z \in \Omega\ \exists k: f^{(k)}(z) \neq 0$). ∎

Let $f \in \mathcal{A}(\Omega)$, $f \not\equiv 0$.

Def. $z_0$ is a zero of $f$ of order $h$ if

$f(z_0) = \dots = f^{(h-1)}(z_0) = 0,\quad f^{(h)}(z_0) \neq 0$

Equivalently: $f(z) = (z-z_0)^h f_1(z)$, $f_1 \in \mathcal{A}(\Omega)$, $f_1(z_0) \neq 0$. ).

Proof. ($\Downarrow$) Let $R := \mathrm{dist}(z_0, \partial\Omega)$. $|z-z_0| < R \Rightarrow f(z) = \sum_{k=h}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$

Let $f_1(z) = \begin{cases} \frac{f(z)}{(z-z_0)^h}, & z \neq z_0 \\ \frac{f^{(h)}(z_0)}{h!}, & z = z_0 \end{cases}$. Then $f_1(z) = \sum_{k=h}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-h}$, $|z-z_0|<R$

So $\lim_{z \to z_0} f_1(z) = f_1(z_0)$

So $f_1 \in \mathcal{A}(\Omega)$.

($\Uparrow$) $f_1(z) = \sum_{k=0}^{\infty} a_{k+h}(z-z_0)^k \Rightarrow f(z) = \sum_{k=h}^{\infty} a_k (z-z_0)^k$,

so $\underset{f(z_0)}{a_0} = \underset{\frac{f'(z_0)}{1!}}{a_1} = \dots = \underset{\frac{f^{(h-1)}(z_0)}{(h-1)!}}{a_{h-1}} = 0$. ∎

Corollary. Let $f \in \mathcal{A}(\Omega)$, $f \not\equiv 0$.

Then $\forall z_0 \in \Omega\ \exists \delta > 0 : 0 < |z-z_0| < \delta \Rightarrow f(z) \neq 0$.

Proof. $f(z_0) \neq 0$ – continuity.

$f(z_0) = 0 \Rightarrow f(z) = (z-z_0)^h f_1(z)$, $f_1(z_0) \neq 0 \Rightarrow \exists \delta>0\ f_1(z) \neq 0\ \forall z: |z-z_0|<\delta$.

So $f(z) = (z-z_0)^h f_1(z) \neq 0\ \forall z: 0<|z-z_0|<\delta$.

Restatement (Uniqueness theorem).

$f \in \mathcal{A}(\Omega)$, $\exists z_n \to z \in \Omega$, $z_n \in \Omega$, $z_n \neq z_m$ : $\forall n\ f(z_n) = 0 \Rightarrow f \equiv 0$.

Proof. $\forall \delta > 0\ \exists z_n \in B(z,\delta) \Rightarrow f \equiv 0$ ∎

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Corollary $f, g \in \mathcal{A}(\mathbb{D})$, $\forall n \in \mathbb{N}: f(\frac{1}{n}) = g(\frac{1}{n}) \Rightarrow \forall z: f(z) = g(z)$.

Case 2. $f \in \mathcal{A}(\Omega \setminus \{z_0\})$, $\lim_{z \to z_0} f(z) = \infty$ — pole of a function $f$.

Consider $g(z) := \frac{1}{f(z)}$. $g \in \mathcal{A}(B(z_0,\delta) \setminus \{z_0\})$ for some $\delta > 0$ (where $f(z) \neq 0$).

$\lim_{z \to z_0} g(z) = 0 \Rightarrow z_0$ is a removable singularity of $g$.

Extend to $g(z_0) = 0$, then $g \in \mathcal{A}(B(z_0,\delta))$, $g(z_0) = 0$.

So $g(z) = (z-z_0)^h g_1(z)$ for some $h \in \mathbb{N}$, $g_1(z) \in \mathcal{A}(B(z_0,\delta))$, $g_1(z_0) \neq 0$. Let $f_1(z) := \frac{1}{g_1(z)}$ $\in \mathcal{A}(B(z,r))$, $r \leq \delta$.

Now $f(z) = \frac{1}{g(z)} = (z-z_0)^{-h} \cdot f_1(z) = \frac{f_1(z)}{(z-z_0)^h}$. $\left(f_1(z_0) = \frac{1}{g_1(z_0)} \neq 0\right)$

$h$ is called an order or multiplicity of the pole.

**Theorem** $z_0$ is a pole of order $h$ iff $f(z) = \frac{a_{-h}}{(z-z_0)^h} + \frac{a_{1-h}}{(z-z_0)^{h-1}} + \dots + \frac{a_{-1}}{z-z_0} + \tilde{f}(z)$ where $\tilde{f}(z) \in \mathcal{A}(B(z_0,r))$ for some $r > 0$, and $a_{-h} \neq 0$.

**Proof.** $(\Rightarrow)$ $f_1(z) = (z-z_0)^h f(z)$ satisfies $f_1(z_0) \neq 0$, $f_1 \in \mathcal{A}(B(z_0,r))$.

So $f_1(z) = \sum b_k (z-z_0)^k \Rightarrow f(z) = (z-z_0)^{-h} f_1(z) = \sum_{k=0}^{h-1} \frac{b_k}{(z-z_0)^{h-k}} + \tilde{f}(z)$,

$b_0 \neq 0$

where $\tilde{f}(z) := \sum_{k=h}^{\infty} b_k (z-z_0)^{k-h} \in \mathcal{A}(B(z_0,r))$.

Take $a_{-k} := b_{h-k}$, then $a_{-h} = b_0 \neq 0$ ∎

$(\Leftarrow)$ $f_1(z) := a_{-h} + a_{1-h}(z-z_0) + \dots + a_{-1}(z-z_0)^{h-1} + \tilde{f}(z)(z-z_0)^h \in \mathcal{A}(B(z_0,r))$

$f_1(z) = (z-z_0)^h f(z)$ $\qquad f_1(z_0) = a_{-h} \neq 0$. ∎

**Def.** $f$ is called meromorphic in a region $\Omega$ if it is analytic in $\Omega$ outside of a set of poles.

**Notation** $\mathcal{M}(\Omega)$.

**Remark** It is a local property: $f \in \mathcal{M}(\Omega) \Leftrightarrow$

$\forall z \in \Omega \; \exists B(z,\delta) : f \in \mathcal{M}(B(z,\delta))$.

**Examples** 1. Let $R(z) = \frac{P(z)}{Q(z)}$ be a rational function.

Then $R \in \mathcal{M}(\mathbb{C})$. Moreover, $R \in \mathcal{M}(\hat{\mathbb{C}})$ (meaning $R(\frac{1}{z}) \in \mathcal{M}(\mathbb{C})$ also).

2. Let $f \in \mathcal{A}(\Omega)$, $g \in \mathcal{A}(\Omega)$, then $\frac{f}{g} \in \mathcal{M}(\Omega)$ if $g \not\equiv 0$.

**Proof.** It is a local property.

Let $z_0 \in \Omega$. $g(z_0) \neq 0 \Rightarrow \exists \delta > 0 \; |z-z_0| < \delta \Rightarrow g(z) \neq 0$

So $\frac{f(z)}{g(z)} \in \mathcal{A}(B(z_0,\delta)) \subset \mathcal{M}(B(z_0,\delta))$

$g(z_0) = 0 \Rightarrow g(z) = (z-z_0)^h g_1(z)$, $g_1(z_0) \neq 0$.

So $\frac{f(z)}{g(z)} = \frac{1}{(z-z_0)^h} \frac{f(z)}{g_1(z)} \in \mathcal{M}(B(z_0,\delta))$ for some $\delta > 0$. ∎

3. If $f \in \mathcal{M}(\Omega)$, $g \in \mathcal{M}(\Omega)$, $g \not\equiv 0$, then $\frac{f}{g} \in \mathcal{M}(\Omega)$.

Same proof as above.

**Def.** Let $z_0$ be a zero or a pole of $f \in \mathcal{M}(\Omega)$. Then for some $h \in \mathbb{Z}\setminus\{0\}$, $f(z)=(z-z_0)^h f_1(z)$, $f_1(z) \in \mathcal{M}(\Omega) \cap \mathcal{A}(B(z_0,r))$, $f_1(z_0)\neq 0$. for some $r>0$. $h$ is called the algebraic order of $f$ at $z_0$.
$h>0$ at a zero, $h<0$ at a pole. Convention: $h=0$ if $f(z_0)\neq 0,\infty$.
Notation: $\operatorname{ord}(f, z_0)$.

## Case 3. Essential singularity.

**Def.** $z_0$ is an essential singularity if it is neither pole nor removable singularity.

Equivalently: 1) $\lim\limits_{z\to z_0} f(z)$ does not exist in $\hat{\mathbb{C}}$.

or 2) there is no $\alpha$ such that $\lim\limits_{z\to z_0} |z-z_0|^\alpha |f(z)|=0$.

Proof. 1) is obviously equivalent.

2) $\Rightarrow$ Essential

If $z_0$ – removable, take $\alpha=1$.
If $z_0$ – pole of order $h$, take any $\alpha>h$:
$f(z)=\dfrac{f_1(z)}{(z-z_0)^h}$, so $\lim\limits_{z\to z_0}|z-z_0|^\alpha |f(z)| = \lim\limits_{z\to z_0}|z-z_0|^{\alpha-h}|f_1(z)| = 0\cdot|f_1(z_0)|=0.$

Essential $\Rightarrow$ 2)

On the other hand, $\lim\limits_{z\to z_0}|z-z_0|^\alpha |f(z)|=0 \Longrightarrow$ take $h\in\mathbb{N}$, $h>0$ and $h>\alpha$.
$\lim\limits_{z\to z_0}|z-z_0|^h|f(z)|=0 \Rightarrow g(z)=(z-z_0)^h f(z)$ has removable singularity at $z_0$
So $f(z)=\dfrac{g(z)}{(z-z_0)^h}$ has a pole of order $\leq h$ (or removable singularity)

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**Theorem** (Sokhotski, Casorati–Weierstrass)
Let $z_0$ be an isolated singularity of $f \in \mathcal{A}(B(z_0,r)\setminus\{z_0\})$
TFAE:
1) $z_0$ is an essential singularity.
2) $\forall\, \delta>0,\ \delta<r,\quad f(B(z_0,\delta)\setminus\{z_0\})$ is dense in $\hat{\mathbb{C}}$,
i.e. $\operatorname{Clos}(f(B(z_0,\delta)\setminus\{z_0\})) = \hat{\mathbb{C}}$

3) $\forall\, w\in\hat{\mathbb{C}},\ \exists\, z_n \to z_0,\ f(z_n)\to w.$

Sokhotski (1868), Casorati (1868), Weierstrass (1876)
Briot & Bouquet (1859, but removed from second edition 1875).

**Proof.** 1) $\Rightarrow$ 2) Since $\mathbb{C}$ is dense in $\hat{\mathbb{C}}$, enough to prove density in $\mathbb{C}$. Take $w\in\mathbb{C}$, assume $w\notin \operatorname{Clos}(f(B(z_0,\delta)\setminus\{z_0\}))$ for some $\delta>0$. Then $\exists\, \varepsilon>0.\ B(w,\varepsilon)\not\ni f(z)\ \forall z\neq z_0.\ |z-z_0|<\delta.$
So $g(z) := \dfrac{1}{f(z)-w} \in \mathcal{A}(B(z_0,\delta)\setminus\{z_0\})$,
$|g(z)| \le \frac{1}{\varepsilon}$. So $g$ is bounded in $B(z_0,\delta)\setminus\{z_0\}$, so
$z_0$ is a removable singularity of $z_0$, $g\in\mathcal{A}(B(z_0,\delta)) \Rightarrow$
$f = \frac{1}{g} + w \in \mathcal{M}(B(z_0,\delta))$ – contradiction.
2) $\Rightarrow$ 3) $\forall\, w\in\hat{\mathbb{C}}\ \forall n\ \exists\, z_n:\ |z_n - z_0| < \frac{1}{n},\ \rho(f(z_n),w)<\frac{1}{n}.$
Then $z_n\to z_0,\ f(z_n)\to w.$
3) $\Rightarrow$ 1) obvious, $\lim\limits_{z\to z_0} f(z)$ does not exist. ∎

Much more is true:
**Theorem** (Great Picard's) If $f\in\mathcal{A}(B(z_0,r)\setminus\{z_0\})$ has an essential singularity at $z_0$, then $\forall\,\delta>0$
$\mathbb{C}\setminus f(B(z_0,\delta)\setminus\{z_0\})$ consists of at most one point.

$e^{\frac{1}{z}}$ has essential singularity at $0$, $e^{\frac{1}{z}}\neq 0$, so it is sharp.

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